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New construction of 3*nj*-symbols

Ya I Granovskii† and A S Zhednov

Physics Department, Donetsk University, Donetsk 340055, Ukraine

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A tribute to the memory of Professor Ya Smorodinski

Abstract. Polynomials, reducing the wavefunction of coupled momenta to products of single ones, are introduced and discussed. Composition of N such polynomials serves as a generating function for $3nj$-coefficients of vector addition. For SU(1, 1) the polynomials are calculated explicitly and a compact integral representation for $9j$-coefficients is obtained as an example. The method is rather general and may be applied to other groups and to higher vector addition coefficients.

The vector addition coefficients ($3nj$-symbols) are defined as unitary matrices transforming wavefunctions from one addition scheme to another. For example, addition of three momenta yields the Racah decomposition

$$|k_1k_2(k_{12})k_3;K\rangle=\sum_{k_{23}}|k_1k_2k_3(k_{23});K\rangle\begin{Bmatrix}k_1 & k_2 & k_{12}\\ k_3 & K & k_{23}\end{Bmatrix}. \tag{1}$$

The main difficulty consists in explicit construction of the wavefunctions corresponding to the given addition scheme. In the commonly accepted way the left-hand and right-hand sides of (1) are expressed via series involving Clebsch–Gordon coefficients [1, 2], leading to cumbersome expressions for $6j$s. Another approach, dealing with the so-called Wigner–Racah algebra [3] is also too awkward.

We propose to use a representation $\langle x|nk\rangle$ for the wavefunction, where $|nk\rangle$ is a standard basis, i.e. eigenfunction of J_0 and Casimir operator J^2

$$J_0|nk\rangle=(n+k)|nk\rangle, \qquad J^2|nk\rangle=k(k-1)|nk\rangle \tag{2}$$

and $|x\rangle$ is an eigenstate of the operator

$$x=\sum_i c_iJ_i.$$

The method proposed is based on the following procedure [5]. As a first step let us factorize $\langle x|nk\rangle$ into so-called 'vacuum' amplitude $\langle x|0k\rangle$ and remainder $Q_n(x;k)$

$$\langle x|nk\rangle=\langle x|0k\rangle Q_n(x;k) \tag{3}$$

which appears to be a classical orthogonal polynomial [4–6].

† Present address: Physics Department, Pedagogical University, 76-200 Slupsk, Poland

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The second step consists in extension of this factorization property to a wavefunction of two particles

$$\langle xy|NK\rangle = \langle xy|0K\rangle Q_N(x+y;K). \tag{4}$$

Here $|x,y\rangle = |x\rangle \otimes |y\rangle$ is an unconnected basis and $|NK\rangle$ is a connected basis in the space of the direct product of two algebras.

The third step is Clebsch–Gordan decomposition of the two-point 'vacuum' amplitude

$$\langle xy|0K\rangle = \langle x|0k_1\rangle\langle y|0k_2\rangle S(x,y;k_1k_2K) \tag{5}$$

where

$$S(x,y;k_1k_2K) = \sum_{m,n} \langle 0K|mk_1, nk_2\rangle Q_m(x;k_1)Q_n(y;k_2) \qquad m+n=K-k_1-k_2 \tag{6}$$

which appears to be a quite definite polynomial determined by the group that is considered.

Taking all this together, we construct the 'vacuum' amplitude for three added momenta in the form

$$\langle xyz|0K\rangle^{(12,3)} = S(x,y;k_1k_2k_{12})S(x+y,z;k_{12}k_3K)\prod_{i=1}^{3}\langle x_i|0k_i\rangle \tag{7}$$

corresponding to $(k_1 \oplus k_2)\oplus k_3$ addition (here $x_1=x$, $x_2=y$, $x_3=z$).

Another possible way of addition $k_1\oplus(k_2\oplus k_3)$ leads to an analogous but different expression

$$\langle xyz|0K\rangle^{(1,23)} = S(y,z;k_2k_3k_{23})S(x,y+z;k_1k_{23}K)\prod_{i=1}^{3}\langle x_i|0k_i\rangle. \tag{8}$$

Connecting (7) and (8) with the help of (1) we obtain

$$S(x,y;k_1k_2k_{12})S(x+y,z;k_{12}k_3K) = \sum_{k_{23}} S(y,z;k_2k_3k_{23})S(x,y+z;k_1k_{23}K)\begin{Bmatrix} k_1 & k_2 & k_{12} \\ k_3 & K & k_{23}\end{Bmatrix} \tag{9}$$

The essential part of this construction is a S-function—a homogeneous polynomial of order $p=K-k_1-k_2$ in arguments x, y. Equation (9) is then a correlation property between products of such polynomials. The absence of any transcendent factor results in considerable simplification.

So far we have not specified the algebra underlying the addition procedure—this may be one of Lie algebras $SU(2)$, $SU(1,1)$, $W(2)$ (Heisenberg–Weyl or oscillator algebra)—that have three generators.

Let us choose for demonstration the SU(1, 1) algebra, leading to a more simple shape of S-function. We restrict ourself to representations of discrete positive series and take $X=2J_0-J_+-J_-$. These choices give Laguerre polynomials for $Q_n(x;k)$ and Jacobi polynomials for $S(x,y;k_1k_2K)$ (details of calculations are given in the appendix).

$$S(x,y;k_1k_2K) = x^p{}_2F_1(-p, 1-2k_1-p; 2k_2|-y/x)\sigma(k_1k_2K)$$

$$p=K-k_1-k_2=0,1,2\ldots \tag{10}$$

Being a polynomial identity, equation (9) is valid for any value of the arguments x, y, z, so giving them particular values, we may deduce from (9) a wide class of useful formulae.

Thus, putting $x=1$, $y=-z$ we get a simple generating function for $6j$-symbols

$$
{}_2F_1(k_1+k_2-k_{12}, 1-k_1+k_2-k_{12}; 2k_2|z)\,{}_2F_1(k_{12}+k_3-K, K+k_{12}+k_3-1; 2k_3|z)
$$

$$
=\sum_{n=0}^{N} G_n Z^n \begin{Bmatrix} k_1 & k_2 & k_{12} \\ k_3 & K & k_2+k_3+n \end{Bmatrix} \tag{11}
$$

where

$$
G_n=(-1)^n \frac{(2k_2+2k_3+n-1)_n \sigma(k_2k_3k_{23})\sigma(k_1k_{23}K)}{(2k_3)_n \sigma(k_1k_2k_{12})\sigma(k_{12}k_3K)}
$$

$$
N=K-k_1-k_2-k_3=0,1,2,\ldots
$$

As a consequence we can immediately obtain from (11) the explicit expression for $6j$-symbols in terms of the generalized hypergeometric function ${}_4F_3(1)$. It is instructive to compare this method of derivation with the traditional one [3].

The method may be applied without complications to addition of four (and more) momenta. The addition scheme $(k_1 \oplus k_2)\oplus(k_3\oplus k_4)$ yields the following factorization of four-momenta 'vacuum' amplitude

$$
\langle xyzu|0K\rangle^{(12,34)} = S(x,y;k_1k_2k_{12})S(z,u;k_4k_4k_{12})
$$

$$
\times S(x+y,z+u;k_{12}k_{34}K)\prod_{i=1}^{4}\langle x_i|0k_i\rangle. \tag{12}
$$

Another addition scheme $(k_1\oplus k_3)\oplus(k_2\oplus k_4)$ yields the factorization

$$
\langle xyzu|0K\rangle^{(13,24)} = S(x,z;k_1k_3k_{13})
$$

$$
\times S(y,u;k_2k_4k_{24})S(x+z,y+u;k_{13}k_{24}K)\prod_{i=1}^{4}\langle x_i|0k_i\rangle. \tag{13}
$$

Coupling (12) and (13) by means of $9j$-symbols gives the identity for ternary products of Jacobi polynomials

$$
S(x,y;k_1k_2k_{12})S(z,u;k_3k_4k_{34})S(x+y,z+u;k_{12}k_{34}K)
$$

$$
=\sum_{k_{13},k_{24}} S(x,z;k_1k_3k_{13})S(y,u;k_2k_4k_{24})S(x+z,y+u;k_{13}k_{24}K)\,\{9j\} \tag{14}
$$

where

$$
\{9j\}=\begin{Bmatrix} k_1 & k_3 & k_{13} \\ k_2 & k_4 & k_{24} \\ k_{12} & k_{34} & K \end{Bmatrix} \tag{15}
$$

is a concise notation for $9j$-symbol.

Among numerous applications of this relation we would like to note the new integral representation of $9j$-symbols. It is obtained by setting $x+y=-(z+u)$ and then using the orthogonality of Jacobi polynomials to get rid of them from the right-hand side of (14). After simple manipulation we obtain

$$\begin{Bmatrix} k_1 & k_3 & k_1+k_3+m \\ k_2 & k_4 & k_2+k_4+n \\ k_{12} & k_{34} & K \end{Bmatrix} = C\int_{-1}^{1} \mathrm{d}v \int_{-1}^{1} \mathrm{d}w (v-w)^N (1-v)^{\alpha_1}(1-w)^{\alpha_2}(1+v)^{\alpha_3}(1+w)^{\alpha_4}$$
$$P^{(\alpha_1,\alpha_2)}\left(\frac{2-v-w}{v-w}\right) P_q^{(\alpha_3,\alpha_4)}\left(\frac{2+v+w}{w-v}\right) P_m^{(\alpha_1,\alpha_3)}(v) P_n^{(\alpha_2,\alpha_4)}(w) \tag{16}$$

where

$$N = K - \sum_{i=1}^{4} k_i \qquad p = k_{12}-k_1-k_2 \qquad q = k_{34}-k_3-k_4 \qquad \alpha_i = 2k_i - 1 \tag{17}$$

and an expression for constant C is given in the appendix.

The integral representation (16) is perhaps the simplest among those known (compare, for example, with [8]).

By means of this method the reader may obtain the formulae for 12*j*-symbols and their triple-integral representation.

In conclusion let us underline that the simplicity of the relations obtained here is tightly bounded with the choice of x-representation (and the explicit form of the operator X) and factorizing the 'vacuum' amplitudes out. Evidently, it turns out to be possible owing to independence of 6*j*-, 9*j*- etc. on the quantum numbers n summed up in S-function.

The late Professor Ya Smorodinskii, a known expert in the field, wrote some years ago '9*j*—that is a true goal to think about'. We would like to hope that equation (14) fulfills all his great demands.

Appendix

Taking the matrix element $\langle x|2J_0 - J_+ - J_-|nk\rangle$, we obtain the relation for Q-polynomials, defined in (3)

$$xQ_n(x;k) = 2(n+k)Q_n(x;k) - a_{n+1}Q_{n+1}(x;k) - a_nQ_{n-1}(x;k). \tag{A1}$$

For the algebra SU(1, 1) from its commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_-, J_+] = 2J_0 \tag{A2}$$

the coefficients

$$a_n = \langle n-1, k|J_-|nk\rangle = \sqrt{n(n+\alpha)} \qquad \alpha = 2k-1, n = 0, 1, 2\ldots \tag{A3}$$

are obtained.

Inserting these coefficients into (A1) and comparing the resulting expression with the recurrence relation for Laguerre polynomials [7] we conclude that

$$Q_n(x;k) = [n!/(\alpha+1)_n]^{1/2} L_n^\alpha(x) \tag{A4}$$

Returning to the calculation of $S(x, y; k_1k_2K)$ via the formula (6), we use the expression for the Clebsch–Gordan coefficient

$$\langle 0K|mk_1, nk_2\rangle = \frac{D_p(-1)^m}{[m!n!(2k_1)_m(2k_2)_n]^{1/2}} \tag{A5}$$

where the normalization factor D_p is equal to

$$D_p = \left[p!\frac{(2k_1)_p(2k_2)_p}{(k_1+k_2+K-1)_p}\right]^{1/2} \qquad p = K - k_1 - k_2$$

$$(a)_n = a(a+1)\ldots(a+n-1). \tag{A6}$$

From (6) and (A4) one obtains

$$S(x, y; k_1k_2K) = D_p \sum_{m+n=p} (-1)^m \frac{L_m^{\alpha_1}(x)L_n^{\alpha_2}(y)}{(\alpha_1+1)_m(\alpha_2+1)_n}. \tag{A7}$$

Substituting the series expansion for $L_m^{(\alpha 1)}$, $L_n^{(\alpha 2)}$ [7] and taking sums over m and n, we arrive at the one-fold sum that is reduced to a Gauss hypergeometric function

$$S(x, y; k_1k_2K) = x^p {}_2F_1(-p, 1-2k_1-p; 2k_2|-y/x)\sigma(k_1k_2K) \tag{A8}$$

where

$$\sigma(k_1k_2K) = \left[\frac{(2k_2)_p}{p!(2k_1)_p(k_1+k_2+K-1)_p}\right]^{1/2} \tag{A9}$$

which is just formula (10).

In the main text we used the following properties of the S-function

$$\begin{aligned} &S(x, y; k_1k_2K) = (-1)^p S(y, x; k_2k_1K) \\ &S(x, 0; k_1k_2K) = x^p\sigma(k_1k_2K) \\ &S(x, -x; k_1k_2K) = x^p A(k_1k_2K) \\ &S(x, y; k_1k_2K) = (x+y)^p P_p^{(\alpha_1, \alpha_2)}\left(\frac{y-x}{y+x}\right)B(k_1k_2K) \end{aligned} \tag{A10}$$

where $P_p^{(\alpha,\beta)}$ are standard Jacobi polynomials [7]. The normalization constants A and B are given as

$$\begin{aligned} &A(k_1k_2k_{12}) = \left[\frac{(2k_1+2k_2+p-1)_p}{p!(2k_1)_p(2k_2)_p}\right]^{1/2} \\ &B(k_1k_2k_{12}) = (-1)^p \frac{p!A(k_1k_2k_{12})}{(2k_1+2k_2+p-1)_p}. \end{aligned} \tag{A11}$$

Finally constant C in (16) is

$$C = (-1)^{n+q} \frac{A(k_{12}k_{34}K)B(k_1k_2k_{12})B(k_3k_4k_{34})}{A(k_{13}k_{24}K)B(k_2k_4k_{24})h_m(\alpha_1, \alpha_3)h_n(\alpha_2, \alpha_4)} \tag{A12}$$

and $h_n^{(\alpha,\beta)}$ are normalization constants for Jacobi polynomials.

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